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Twisted K-theory in $g > 1$ from D-branes

José M. Isidro

Department of Theoretical Physics, 1 Keble Road, Oxford OX1 3NP, UK

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Abstract

We study the wrapping of N-type IIB Dp-branes on a compact Riemann surface Σ in genus $g > 1$ by means of the Sen–Witten construction, as a superposition of N'-type IIB Dp'-brane/antibrane pairs, with $p' > p$. A background Neveu–Schwarz field B deforms the commutative $C^{\star}$-algebra of functions on Σ to a non-commutative $C^{\star}$-algebra. Our construction provides an explicit example of the $N' \to \infty$ limit advocated by Bouwknegt-Mathai and Witten in order to deal with twisted K-theory. We provide the necessary elements to formulate M(atrix) theory on this new $C^{\star}$-algebra, by explicitly constructing a family of projective $C^{\star}$-modules admitting constant-curvature connections. This allows us to define the $g > 1$ analogue of the BPS spectrum of states in $g = 1$, by means of Donaldson's formulation of the Narasimhan–Seshadri theorem. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

1.1. Setting

The fact that D-branes carry vector bundles has allowed to interpret D-brane charges and fields as classes in the K-theory of space–time, rather than as integer cohomology classes [1–9]. This identification has led to a better understanding of the spectrum of D-branes, in particular of stable, non-supersymmetric D-branes. Such non-BPS branes can often be understood as bound states of a brane–antibrane system with tachyon condensation [10].

E-mail address: isidro@thphys.ox.ac.uk (J.M. Isidro).

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It has been proposed [6,11] that the K-theory analysis of a superposition of N'-type IIB Dp'-brane/antibrane pairs is best performed in the limit $N' \to \infty$. This limit allows for the possibility of considering a non-torsion class for the field strength $H = \mathrm{d}B$ of the Neveu–Schwarz field B.

Along a related line, M(atrix) theory [12,13] as a model for M-theory has been compactified toroidally in [14]. By turning on a background B-field one can deform this compactification to a compactification on the non-commutative torus [15,16]. The effective gauge theory on the D-branes then becomes non-commutative [17].

1.2. Aim

In this paper, we combine the three lines named above. The aim is to provide a physical interpretation for the $C^{\star}$-algebra constructed abstractly in [18]. The strategy is as follows.

We first wrap N-type IIB Dp-branes on a manifold $\Sigma \times Y$, where Σ is compact Riemann surface with genus $g > 1$ and Y is an auxiliary space–time manifold to be specified presently. (With more generality, one could consider a nontrivial bundle over Σ instead of $\Sigma \times Y$.) Following [10], each one of the N wrapped Dp-branes can be viewed as a superposition of N' Dp'-brane/antibrane pairs, with an odd value of $p' > p$. When wrapping a single-type IIB Dp-brane on a manifold of codimension $2k$, a minimum of $N' = 2^{k-1}$-type IIB Dp'-brane/antibrane pairs are needed [2]. Eventually passing to the limit $N \to \infty$ will also enforce $N' \to \infty$ and bring us into the stable range of K-theory. Simultaneously we turn on a background B-field across Σ.

On the other hand, this system possesses a dual description in type IIA string theory or, more precisely, in 11-dimensional M-theory as described by M(atrix) theory. In this dual setting, a Dp-brane is compactified on p copies of S^1, next T-dualised along all p circles, and finally decompactified into a type IIA D0-brane. The limit $N \to \infty$ required by M(atrix) theory has a natural counterpart in the dual-type IIB description: it arises from the requirement of allowing for the possibility that the background field strength H be a non-torsion class. Our model provides an explicit realisation, in string theory terms, of the twisted K-theory described abstractly by Bouwknegt and Mathai [11], and advocated by Witten [6] in a similar K-theoretic setting.

From this M(atrix) theory, description of the wrapped Dp-branes, the connection with non-commutative geometry [19] is now immediate: the background B-field deforms the commutative $C^{\star}$-algebra of functions on Σ to a non-commutative $C^{\star}$-algebra.

1.3. Outline

This paper is organised as follows. As a preparation for $g > 1$, Section 2 reviews the non-commutative torus from the standpoint of the Heisenberg algebra. The latter can be interpreted as a *central-curvature condition* on a projective module over the non-commutative torus [20,21]. (*Central* means that as an endomorphism of the projective module, the curvature is proportional to the identity. By abuse of terminology, we will call Eq. (16) below a *constant-curvature condition*, rather than a central-curvature condition.)

The constant-curvature condition has a natural extension to $g > 1$ in the theory of stable, holomorphic vector bundles over a Riemann surface Σ, together with Donaldson's version

[22] of the Narasimhan–Seshadri theorem [23]. The latter provides the right mathematical description of the twisted gauge bundles arising on the stack of coincident branes required by the Sen–Witten construction of non-BPS branes. Indeed such bundles can be characterised as admitting a constant-curvature connection. These points are summarised in Section 3.

Section 4 presents this new $C^{\star}$-algebra. We recall from [18] the definition of its generators and of the trace required to write down the M(atrix) theory action, and explicitly construct the corresponding projective $C^{\star}$-modules.

Wrapping a Dp-brane on a closed, $(p+1)$-dimensional submanifold of space–time is possible only when the condition of cancellation of global worldsheet anomalies is satisfied [2,24–26]. This point is dealt with in Section 5. In particular, this analysis fixes the dimensionality of the Dp-branes to be $p \geq 3$; this bound will be later refined by cohomological arguments in Section 7.4. Section 6 presents, following [11,26], the necessary formalism about the background field strength, oriented towards the limit $N \to \infty$ that will be taken in Section 7.

In Section 7, we first describe the setup in type IIB string theory terms. Next we pass, through a duality transformation, to an equivalent M(atrix) theory description of the N Dp-branes wrapped on Σ. As $N \to \infty$, so too must the 't Hooft magnetic flux M go to infinity, in a certain sense to be specified presently. We will analyse this *double scaling limit* in detail; our $C^{\star}$-algebra of [18] is precisely the double scaling limit of the Narasimhan–Seshadri representations of the Fuchsian group Γ uniformising the Riemann surface Σ in $g > 1$.

In Section 8, we use Donaldson's theorem to identify the $g > 1$ analogues of BPS states on the non-commutative torus, by explicitly identifying constant-curvature connections on the projective $C^{\star}$-modules constructed in Section 4. Finally, Section 9 presents some conclusions and perspectives.

2. BPS spectra in $g=1$ from the Stone–von Neumann theorem

2.1. The constant-curvature condition

Let us set the fermions of the M(atrix) theory action to zero, and consider a state determined by the condition that a connection on a projective module over the non-commutative torus T^2_θ have constant field strength

$$F_{jk} = \omega_{jk}\mathbf{I}, \tag{1}$$

i.e. the curvature must be proportional to the identity endomorphism. Above, ω_{jk} is a constant 2-form over the Lie algebra of derivations of T^2_θ. In the presence of supersymmetry such field configurations give rise to BPS states [15,20], with an amount of preserved supersymmetry given by the dimension of the space of spinors ϵ, ϵ' that solve the equation

$$\epsilon\Gamma^{jk}F_{jk} + \epsilon'\mathbf{I} = 0, \tag{2}$$

where $\Gamma^{jk} = [\Gamma^j, \Gamma^k]$ is a commutator of Dirac matrices. In the absence of supersymmetry, as will be the case in $g > 1$, condition (1) is the closest analogue of Eq. (2) defining a BPS state.

In [21], it has been argued that the complete set of equations specifying a projective module over the torus T^2_θ, together with a constant-curvature connection on it, is given by

$$\mathcal{U}_j\mathcal{U}_k = \mathrm{e}^{2\pi \mathrm{i}\theta_{jk}}\mathcal{U}_k\mathcal{U}_j, \qquad [\nabla_j, \mathcal{U}_k] = \delta_{jk}\mathcal{U}_k, \qquad [\nabla_j, \nabla_k] = \mathrm{i}F_{jk}\mathbf{I}, \tag{3}$$

where $j, k = 1, 2$. These equations can be solved by first representing the Heisenberg algebra $[\nabla_j, \nabla_k] = \mathrm{i}F_{jk}\mathbf{I}$, through the Stone–von Neumann theorem, on the Hilbert space $L^2(\mathbf{R})$. The hermiticity of this representation ensures the unitarity of the generators

$$\mathcal{U}_j = \exp(\mathrm{i}F_{jk}^{-1}\nabla_k). \tag{4}$$

Next we tensor the latter with an $(N \times N)$-dimensional representation of 't Hooft's matrices u_j

$$u_j u_k = \mathrm{e}^{2\pi \mathrm{i}M_{jk}/N} u_k u_j, \quad M_{jk} \in \mathbf{Z}, \tag{5}$$

acting on the space $\mathbf{C}^N$. The complete projective module over T^2_θ is given by

$$E_{NM} = L^2(\mathbf{R}) \otimes \mathbf{C}^N_{(M)}, \tag{6}$$

where the notation $\mathbf{C}^N_{(M)}$ makes reference to the magnetic flux $M = M_{12}$. The total generators

$$\mathcal{U}_j \otimes u_j, \quad j = 1, 2 \tag{7}$$

satisfy the algebra of T^2_θ with a total deformation parameter

$$\theta_{jk} = -\frac{1}{2\pi}F_{jk} + \frac{1}{N}M_{jk}. \tag{8}$$

The fact that $F_{jk}\mathbf{I}$ is a c-number allows one to compute the deformation parameter by a simple application of the Baker–Campbell–Hausdorff formula.

2.2. Moduli space of constant-curvature connections

The notion of a moduli space $\mathcal{M}^{(g=1)}$ of constant-curvature connections in $g = 1$ appears naturally in the above picture [21]. $\mathcal{M}^{(g=1)}$ is the space of solutions to the first two equations of (3), modulo gauge transformations. Modules possessing different Chern numbers are treated simultaneously in this approach. Fixing a Chern number corresponds to choosing a connected component of the total moduli space of solutions to Eq. (3).

The residual gauge transformations preserving the constant-curvature condition (3) correspond to $N \times N$ unitary transformations acting on the $\mathbf{C}^N$ factor of the module E_{NM}. Hence the moduli space of constant-curvature connections on a module with fixed integer values of N and M_{jk} can be described as a space of inequivalent representations of the matrix algebra (5). The latter in fact admits a continuum of inequivalent representations. In order to identify it, we first consider the commutative torus $\hat{T}^2$ that is dual to the original commutative torus T^2. Then the space of irreducible representations of (5) is described by means of two complex numbers λ_i with unit modulus, modulo a certain residual gauge symmetry. Let E_Λ, $\Lambda = (\lambda_1, \lambda_2)$, denote the corresponding irreducible representations, and assume that $\mathbf{C}^N$ decomposes as $\mathbf{C}^N = \oplus_{l=1}^r E_{\Lambda_l}$. The residual gauge symmetry acts by permutation on the r summands as the permutation group S_r, and the moduli space $\mathcal{M}^{(g=1)}$ is $\hat{T}^2/S_r$.

3. The Narasimhan–Seshadri theorem

3.1. Statement of the theorem

Let Γ denote the Fuchsian group uniformising a compact Riemann surface Σ with genus $g > 1$ and without boundary. We now summarise some facts about projective, unitary representations of Γ and the theory of holomorphic vector bundles over Σ [23] (for more extensive treatments see [27,28]).

Let $\mathcal{E} \to \Sigma$ be a holomorphic vector bundle over Σ of rank N and degree, i.e. first Chern class, M. The bundle $\mathcal{E}$ is called *stable* if the ratio

$$\mu(\mathcal{E}) = \frac{M}{N} \tag{9}$$

satisfies the inequality $\mu(\mathcal{E}') < \mu(\mathcal{E})$ for every proper holomorphic sub-bundle $\mathcal{E}' \subset \mathcal{E}$. We may take $-N < M \leq 0$, as this may always be arranged by tensor multiplication with a line bundle without losing stability.

Denote by γ_j, $j = 1, \dots, 2g$, the generators of Γ. We have

$$\prod_{j=1}^{g} (\gamma_{2j-1}\gamma_{2j}\gamma_{2j-1}^{-1}\gamma_{2j}^{-1}) = \mathbf{I}. \tag{10}$$

For the purposes of this section we will temporarily assume that Γ contains a unique primitive elliptic element γ_0 of order N, i.e. $\gamma_0^N = \mathbf{I}$, with fixed point $z_0 \in \mathbf{H}$ that projects to $x_0 \in \Sigma$. Now let $\rho : \Gamma \to U(N)$ be an irreducible unitary representation. It is said *admissible* if

$$\rho(\gamma_0) = \mathrm{e}^{-2\pi \mathrm{i} M/N}\mathbf{I}. \tag{11}$$

Putting the elliptic element on the right-hand side, and denoting $\rho(\gamma_j)$ by u_j, an admissible representation satisfies

$$\prod_{j=1}^{g} (u_{2j-1}u_{2j}u_{2j-1}^{-1}u_{2j}^{-1}) = \mathrm{e}^{2\pi \mathrm{i} M/N}\mathbf{I}. \tag{12}$$

The u_j are the $g > 1$ generalisation of ’t Hooft’s matrices (5).

On the trivial bundle $\mathbf{H} \times \mathbf{C}^N \to \mathbf{H}$ there is an action of $\Gamma : (z, v) \mapsto (\gamma z, \rho(\gamma)v)$. This defines the quotient

$$\frac{\mathbf{H} \times \mathbf{C}^N}{\Gamma} \to \frac{\mathbf{H}}{\Gamma} \cong \Sigma. \tag{13}$$

Any admissible representation determines a holomorphic vector bundle $\mathcal{E}_\rho \to \Sigma$ of rank N and degree M. When $M = 0$, $\mathcal{E}_\rho$ is simply the quotient bundle (13). The Narasimhan–

Seshadri theorem now states that a holomorphic vector bundle $\mathcal{E} \to \Sigma$ of rank N and degree M is stable if and only if it is isomorphic to a bundle $\mathcal{E}_\rho$, where ρ is an admissible representation of Γ. Moreover, the bundles $\mathcal{E}_{\rho_1}$ and $\mathcal{E}_{\rho_2}$ are isomorphic if and only if the representations ρ_1 and ρ_2 are equivalent.

Next consider the adjoint representation of Γ on $\mathrm{End}\,\mathbf{C}^N$,

$$\mathrm{Ad}\,\rho(\gamma)Z = \rho(\gamma)Z\rho^{-1}(\gamma), \tag{14}$$

where $Z \in \mathrm{End}\,\mathbf{C}^N$ is understood as an $N \times N$ matrix. Let us also consider the trivial bundle $\mathbf{H} \times \mathrm{End}\,\mathbf{C}^N \to \mathbf{H}$. There is an action of $\Gamma : (z, Z) \mapsto (\gamma z, \mathrm{Ad}\,\rho(\gamma)Z)$ that defines the quotient bundle

$$\frac{\mathbf{H} \times \mathrm{End}\,\mathbf{C}^N}{\Gamma} \to \frac{\mathbf{H}}{\Gamma} \cong \Sigma. \tag{15}$$

When $\mathcal{E}$ is stable, the bundle of endomorphisms $\mathrm{End}\,\mathcal{E} \to \Sigma$ is isomorphic to the quotient bundle (15).

3.2. *Donaldson's approach to stability of vector bundles*

A differential-geometric approach to stability has been given by Donaldson [22]. Fix a Hermitian metric on Σ, e.g. the Poincaré metric, normalised so that the area of Σ equals 1. Let us denote by ω its associated (1, 1)-form. Then a holomorphic vector bundle is stable if and only if it admits a metric connection ∇_D with constant-curvature

$$F_D = -2\pi \mathrm{i}\mu(\mathcal{E})\omega\mathbf{I}, \tag{16}$$

and such a connection ∇_D is unique. As done for BPS states in $g = 1$ [20], in Section 8 we will use the constant-curvature condition (16) to characterise BPS-like states in $g > 1$.

4. Infinite-dimensional projective representations of the Fuchsian group Γ

In order to study M(atrix) theory in $g > 1$, the following elements are needed: a knowledge of the $C^\star$-algebra, a trace and the projective $C^\star$-modules.

4.1. *Definition of the $C^\star$-algebra $C^\star(\Gamma, \theta)$*

Let us recall from [18] the construction of the operators $\mathcal{U}_j = \rho_b(\gamma_j)$ that provide a projectively unitary representation ρ_b of the Fuchsian group Γ. We first pick a fundamental domain $\mathcal{F}_z$ for the Fuchsian group Γ uniformising Σ, with basepoint $z \in \mathbf{H}$, in order to have a tessellation $T(\mathbf{H})$ of $\mathbf{H}$. On the Hilbert space $L^2(\mathbf{H})$ one defines, for every value of the Fuchsian index $j = 1, \ldots, 2g$,

$$\mathcal{U}_j^{(z)} = \exp\left(\mathrm{i}b\int_z^{\gamma_j z} A\right) \prod_{\alpha=-1}^{1} \exp[\lambda_\alpha^{(j)}(L_\alpha + \bar{L}_\alpha)]. \tag{17}$$

Above, the $L_\alpha, \bar{L}_\alpha$ are the standard $\mathrm{sl}_2(\mathbf{R})$ differential generators $z^{\alpha+1}\partial_z$, $\bar{z}^{\alpha+1}\partial_{\bar{z}}$, $A = \mathrm{d}\,\mathrm{Re}(z)/\mathrm{Im}(z)$ is a gauge field on $\mathbf{H}$, the $\lambda_\alpha^{(j)}$ are a set of numerical parameters specifying a complex structure on Σ, and b is an arbitrary real parameter. One can prove that the $\mathcal{U}_j^{(z)}$ are unitary and satisfy

$$\prod_{j=1}^{g}(\mathcal{U}_{2j-1}\mathcal{U}_{2j}\mathcal{U}_{2j-1}^{-1}\mathcal{U}_{2j}^{-1}) = \mathrm{e}^{-2\pi \mathrm{i}\theta_b}\mathbf{I} \tag{18}$$

with θ_b a real parameter that is independent of the basepoint z and is given by

$$\theta_b = b\chi(\Sigma) = b(2-2g). \tag{19}$$

Consider the associative algebra with involution whose unitary generators are the $\mathcal{U}_j^{(z)}$ of Eq. (18). It admits a faithful unitary representation on $L^2(\mathbf{H})$. Taking the norm closure of this image [19], this algebra becomes a $C^\star$-algebra that we denote by $C^\star(\Gamma, \theta)$.

4.2. *Definition of the trace*

A trace can be defined by means of the following equivalent presentation of $C^\star(\Gamma, \theta)$ [18]. Each $\gamma \neq \mathbf{I}$ in Γ can be univocally expressed as a positive power of a primitive element $\tilde{p} \in \Gamma$, *primitive* meaning that it is not a positive power of any other element in Γ [29]. Let $\mathcal{V}_{\tilde{p}}$ be the representative of $\tilde{p}$. Any $\mathcal{V} \in C^\star$ can be written as

$$\mathcal{V} = \sum_{\tilde{p}\in\{\mathrm{prim}\}}\sum_{n=0}^{\infty} c_n^{(\tilde{p})}\mathcal{V}_{\tilde{p}}^n + c_0\mathbf{I} \tag{20}$$

for certain coefficients $c_n^{(\tilde{p})}$, c_0. We now define a trace as

$$\mathrm{tr}\,\mathcal{V} = c_0. \tag{21}$$

4.3. *Construction of projective $C^\star(\Gamma, \theta)$-modules E_{NM}*

The Hilbert space $L^2(\mathbf{H})$ becomes a right $C^\star(\Gamma, \theta)$-module under right multiplication of $\xi \in L^2(\mathbf{H})$ with the $\mathcal{U}_j^{(z)}$. A $C^\star(\Gamma, \theta)$-valued inner product $\langle , \rangle$ on this module can be defined by summing over the Fuchsian indices, and over the vertices $z \in T(\mathbf{H})$:

$$\langle \xi, \eta\rangle = \sum_{z\in T(\mathbf{H})}\sum_{j=1}^{2g}(\xi, \eta\mathcal{U}_j^{(z)\dagger})\mathcal{U}_j^{(z)}, \quad \xi, \eta \in L^2(\mathbf{H}). \tag{22}$$

In Eq. (22), $(,)$ denotes the Hermitian product on $L^2(\mathbf{H})$ constructed with respect to the Poincaré metric on $\mathbf{H}$. Next we tensor the differential operators $\mathcal{U}_j^{(z)}$ with a set of Narasimhan–Seshadri matrices u_j. A projective $C^\star(\Gamma, \theta)$-module E_{NM} is defined as the tensor product of $L^2(\mathbf{H})$ times the Narasimhan–Seshadri representation space $\mathbf{C}^N_{(M)}$ with degree M:

$$E_{NM} = L^2(\mathbf{H}) \otimes \mathbf{C}^N_{(M)}. \tag{23}$$

The total generators on E_{NM} are $\mathcal{U}_j^{(z)} \otimes u_j$, with the matrix part contributing a piece

$$\langle \xi_N, \eta_N \rangle_N = \sum_{j=1}^{2g} (\xi_N, \eta_N u_j^{\dagger}) u_j, \quad \xi_N, \eta_N \in \mathbf{C}^N \tag{24}$$

to the scalar product on E_{NM}. In Eq. (24), (,) denotes the standard Hermitian product on $\mathbf{C}^N$. The total deformation parameter for the generators $\mathcal{U}_j^{(z)} \otimes u_j$ is then

$$\theta_{\text{tot}} = \theta_b - \frac{M}{N}. \tag{25}$$

5. The anomaly-cancellation condition

In type IIB superstring theory on a space–time X, consider wrapping a Dp-brane on a closed, $(p+1)$-dimensional submanifold $Q \subset X$. The analysis of global worldsheet anomalies for open superstrings attached to Dp-branes has been performed in [2,25,26]. Let us briefly summarise it.

In the presence of a background Neveu–Schwarz 2-form B, a single Dp-brane can be wrapped on a submanifold $Q \subset X$ if and only if the normal bundle $\mathcal{N}$ of Q satisfies the condition of cancellation of global anomalies for open superstrings ending on Q:

$$\beta_2(w_2(\mathcal{N})) = [H]_Q. \tag{26}$$

Here $[H]$ is the integer cohomology class whose de Rham representative is $H = \mathrm{d}B$, $[H]_Q$ denotes its restriction to Q, and $\beta_2(w_2(\mathcal{N}))$ is the image of the second Stiefel–Whitney class $w_2(\mathcal{N}) \in H^2(Q, \mathbf{Z}_2)$ under the Bockstein homomorphism $\beta_2 : H^2(Q, \mathbf{Z}_2) \to H^3(Q, \mathbf{Z})$ induced by the short exact sequence

$$0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}_2 \to 0. \tag{27}$$

Above, the second arrow is multiplication by 2, while the third arrow is reduction modulo 2.

The wrapping of N Dp-branes on a submanifold Q is governed by a generalisation of Eq. (26) that we describe next. When $[H]_Q = 0$, the N Dp-branes carry an $U(N)$ principal bundle while, for $[H]_Q \neq 0$, the Dp-branes carry a principal $SU(N)/\mathbf{Z}_N$ bundle that cannot be lifted to a $U(N)$ bundle. Now the 't Hooft magnetic 2-form is a cohomology class $[f] \in H^2(Q, \mathbf{Z}_N)$. Consider the image of $[f]$ under the Bockstein homomorphism $\beta_N : H^2(Q, \mathbf{Z}_N) \to H^3(Q, \mathbf{Z})$ induced by the short exact sequence

$$0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}_N \to 0, \tag{28}$$

where the second arrow is multiplication by N, while the third arrow is reduction modulo N. The image $\beta_N([f]) \in H^3(Q, \mathbf{Z})$ measures the obstruction to lifting an $SU(N)/\mathbf{Z}_N$ bundle to a $U(N)$ bundle. It turns out that global worldsheet anomalies for open superstrings ending on the Dp-branes cancel if and only if

$$\beta_N([f]) + \beta_2(w_2(\mathcal{N})) = [H]_Q. \tag{29}$$

For the above condition to be nonempty it is required that $p \geq 3$.

6. The background field strength

6.1. Local description of a twisted bundle

An $SU(N)/\mathbf{Z}_N$ bundle without $U(N)$ structure has the following description in terms of transition functions. Take a good covering of X by open sets W_i, and denote by $su(N)$ the Lie algebra of $SU(N)/\mathbf{Z}_N$. A vector bundle associated with the principal $SU(N)/\mathbf{Z}_N$ bundle has sections $f_i : W_i \to su(N)$. Transition functions $g_{ij} : W_i \cap W_j \to U(N)$ are defined on double overlaps such that

$$f_i = g_{ij} f_j g_{ij}^{-1} = g_{ij} f_j g_{ji}, \tag{30}$$

while on triple overlaps $W_i \cap W_j \cap W_k$ the consistency condition

$$g_{ij} g_{jk} g_{ki} = h_{ijk} \tag{31}$$

must be satisfied. Above, h_{ijk} is an Nth root of unity obeying the cocycle relation

$$h_{ijk} h_{ikl} = h_{jkl} h_{ijl} \tag{32}$$

on quadruple overlaps. From here

$$\ln h_{ijk} + \ln h_{ikl} - \ln h_{jkl} - \ln h_{ijl} = 2\pi \mathrm{i} \kappa_{ijkl}, \tag{33}$$

where κ_{ijkl} defines an element $\kappa \in H^3(X, \mathbf{Z})$ which is the obstruction to lifting the $SU(N)/\mathbf{Z}_N$ bundle to a $U(N)$ bundle.

Therefore, in the presence of $[H] \neq 0$, gauge bundles on the branes are described by transition functions that obey Eq. (31). The direct sum of two such twisted bundles obeys the same condition. Under the usual equivalence relation of K-theory, equivalence classes of twisted bundles define the twisted K-theory of X, denoted $K_{[H]}(X)$ [2].

6.2. The Brauer group

The background field strength H determines a class in the Čech cohomology group $H^3(X, \mathbf{Z})$ [30]. The latter decomposes as

$$H^3(X, \mathbf{Z}) = \mathbf{Z} \oplus \cdots \oplus \mathbf{Z} \oplus \mathbf{Z}_{q_1} \oplus \cdots \oplus \mathbf{Z}_{q_s}. \tag{34}$$

The $\mathbf{Z}_q$ pieces are called *torsion subgroups*. Torsion classes determine a subgroup of $H^3(X, \mathbf{Z})$, called the Brauer group of X and denoted $\mathrm{Br}(X)$. Next we give two different descriptions of the latter. One is in terms of finite-dimensional Azumaya algebras over X, the other one is through $\mathcal{K}$-bundles with structure group $\mathrm{Aut}(\mathcal{K})$. The link between these two descriptions of $\mathrm{Br}(X)$ is explained in [11].

6.3. Azumaya algebras over X

Let $M_N(\mathbf{C})$ denote the algebra of complex $N \times N$ matrices. Its automorphism group $\mathrm{Aut}(M_N(\mathbf{C}))$ is $PU(N) = SU(N)/\mathbf{Z}_N$, where $PU(N) = U(N)/U(1)$ denotes the projective unitary group on $\mathbf{C}^N$.

An *Azumaya algebra over* X is a fibre bundle over X, whose typical fibre is the algebra $M_N(\mathbf{C})$. Sections f_i are $M_N(\mathbf{C})$-valued and transition functions g_{ij} are $PU(N)$-valued, in such a way that Eqs. (30)–(33) are satisfied.

For any torsion class $[H] \in H^3(X, \mathbf{Z})$ there is a unique (equivalence class of) Azumaya algebras and the corresponding twisted K-theory, $K_{[H]}(X)$ [11,26].

6.4. $\mathcal{K}$-bundles over X

In the $C^{\star}$-norm topology, the limit [31]

$$\lim_{N\to\infty} M_N(\mathbf{C}) = \mathcal{K} \tag{35}$$

defines the $C^{\star}$-algebra $\mathcal{K}$ of compact operators on an infinite-dimensional, separable Hilbert space $\mathcal{H}$. Let $U(\mathcal{H})$ denote the group of unitary operators on $\mathcal{H}$, and let $PU(\mathcal{H}) = U(\mathcal{H})/U(1)$ be the projective unitary group on $\mathcal{H}$. By the same token we can set

$$\lim_{N\to\infty} \frac{SU(N)}{\mathbf{Z}_N} = PU(\mathcal{H}). \tag{36}$$

Furthermore, it holds that $\mathrm{Aut}(\mathcal{K}) = PU(\mathcal{H})$.

Let us consider a locally trivial bundle $\mathcal{E}$ over X with fibre $\mathcal{K}$ and structure group $\mathrm{Aut}(\mathcal{K})$. Such a bundle is also determined by Eqs. (30)–(33), where now the typical fibre is the algebra $\mathcal{K}$, hence sections f_i are $\mathcal{K}$-valued and transition functions g_{ij} are $PU(\mathcal{H})$-valued [11].

6.5. $H^3(X, \mathbf{Z})$ as parameter space for $\mathcal{K}$-bundles

Isomorphism classes of locally trivial bundles $\mathcal{E}$ over X with fibre $\mathcal{K}$ and structure group $\mathrm{Aut}(\mathcal{K})$ are parameterised by $H^3(X, \mathbf{Z})$. With every *torsion class* in $H^3(X, \mathbf{Z})$ there is associated an isomorphism class of *projectively flat* bundles $\mathcal{E}$ with fibre $\mathcal{K}$ and structure group $\mathrm{Aut}(\mathcal{K})$ [11]. Such bundles are given by a representation of $\pi_1(X)$ into $\mathrm{Aut}(\mathcal{K})$ [27].

The cohomology class in $H^3(X, \mathbf{Z})$ corresponding to a bundle $\mathcal{E}$ with fibre $\mathcal{K}$ and structure group $\mathrm{Aut}(\mathcal{K})$ is called the *Dixmier–Douady invariant* of $\mathcal{E}$; it is denoted $\delta(\mathcal{E})$ [32]. In terms of transition functions, $\delta(\mathcal{E})$ equals the cohomology class κ given in Eq. (33), with the obvious replacements.

7. Wrapping D-branes on a $g > 1$ Riemann surface

7.1. The type IIB description

In what follows we take Q to be a manifold of the form $\Sigma \times Y$, for some (as yet) unspecified manifold Y. We want to wrap N coincident type IIB Dp-branes on Q. Forgetting about the manifold Y for the moment, we will speak of N coincident Dp-branes wrapping Σ.

Now each one of those Dp-branes, through the Sen–Witten construction [2,10], can be thought of as a superposition of $N' = 2^{k-1}$ Dp'-brane/antibrane pairs on $\mathbf{R}^{p+1}$. Here $2k$ is the codimension of the Dp-branes and $p' > p$. According to Sen-Witten [2,10], an appropriate choice for the tachyon field makes this superposition equivalent to a Dp-brane wrapped on Σ. Eventually passing to the limit $N \to \infty$ will also enforce $N' \to \infty$, thus bringing us into the stable range of K-theory. This is in nice agreement with [6,11], where it has been proposed that the K-theory analysis of a superposition of N' Dp'-brane/antibrane pairs is best performed in the limit $N' \to \infty$.

7.2. The dual description: M(atrix) theory

Our system of N Dp-branes wrapped on Σ has a dual description that allows us to make contact with the setup of [18]. We first unwrap the Dp-branes into flat space. Next we compactify them along p spatial coordinates, on p copies of S^1. A further step is to apply a T-duality on all p circles. Finally, we decompactify them by sending their radii to infinity. The result is a system of N D0-branes. So far the Riemann surface Σ has played a spectator role. However, the N D0-branes can be compactified on the original Σ. The resulting system is best understood in 11-dimensional M(atrix) theory compactified on the Riemann surface Σ, as done in [18]. For the rest of this paper, we will adhere to this dual picture. Then the limit $N \to \infty$ required by M(atrix) theory corresponds, in the dual-type IIB description, to considering the $\mathcal{K}$-bundles of [11], rather than the Azumaya algebras of [26].

Some comments are in order. Assume applying $p-1$ T-dualities instead of p, to get a system of D1-branes. The D1-brane is S-dual to the fundamental-type IIB string. The latter can be wrapped on Σ at the cost of breaking all supersymmetry [33]. Hence the Dp-brane wrapped on Σ breaks all supersymmetry, too, and it corresponds to a non-BPS configuration.

The D1-brane can be viewed as the strong-coupling limit of the fundamental type IIB string in 10 dimensions. On the other hand, 11-dimensional M(atrix) theory is a model for M-theory, i.e. for the strong-coupling limit of type IIA string theory. Moreover, T-duality being a perturbative symmetry, it will not exchange the weak and the strong-coupling regimes. This accounts for the mismatch of dimensions between the two dual descriptions we have given.

7.3. The limit $N \to \infty$

By Eq. (29), we have specified a class $[H]_Q$. In the limit $N \to \infty$, this $[H]_Q$ specifies an isomorphism class of $\mathcal{K}$-bundles over Q. Picking a torsion class in $H^3(Q, \mathbf{Z})$ amounts to picking an isomorphism class of projectively flat bundles $\mathcal{E} \to Q$ with fibre $\mathcal{K}$ and structure group $PU(\mathcal{H})$. If we now choose the manifold Y as explained in Section 7.4, then such an isomorphism class of bundles is specified by a representation of $\pi_1(\Sigma)$ into $PU(\mathcal{H})$.

As summarised in Section 4.1, in [18] we have explicitly constructed, on the separable Hilbert space $\mathcal{H} = L^2(\mathbf{H})$, a 1-parameter family $\rho_b, b \in \mathbf{R}$, of projectively unitary representations of the Fuchsian group $\Gamma \simeq \pi_1(\Sigma)$ uniformising Σ. Although infinite-dimensional, these representations ρ_b can be understood as the double-scaling limit $M \to -\infty, N \to \infty$, of the Narasimhan–Seshadri representations ρ_{NM} reviewed in Section 3. The latter represent $\pi_1(\Sigma)$ on $\mathbf{C}^N$, where N is the rank of the gauge group $U(N)$ carried by the stack of N coincident branes, and $M \in \mathbf{Z}$ is the 't Hooft magnetic flux obtained integrating the 't Hooft

2-form $[f]$ over Σ. The parameter $b \in \mathbf{R}$ on which ρ_b depends can be fine-tuned at will. The identification between our ρ_b of Eq. (18), and its finite-dimensional counterpart ρ_{NM} of Narasimhan–Seshadri, Eq. (12), proceeds as follows. The $N \times N$ unitary matrices u_j acting on $\mathbf{C}^N$ become unitary operators $\mathcal{U}_j$ acting on $L^2(\mathbf{H})$,

$$\lim_{N\to\infty, M\to-\infty} u_j = \mathcal{U}_j, \tag{37}$$

and the phase multiplying the identity on the right-hand side of (12) is identified with that on the right-hand side of (18),

$$\lim_{N\to\infty, M\to-\infty} \exp\left(2\pi \mathrm{i}\frac{M}{N}\right) = \exp(-2\pi \mathrm{i}\theta_b). \tag{38}$$

In this way we have determined a 1-parameter family of projectively flat $\mathcal{K}$-bundles $\mathcal{E}_b \to \Sigma$. We conclude that our infinite-dimensional representations ρ_b of $\pi_1(\Sigma)$ of Eq. (18) are induced by turning on a 't Hooft magnetic flux across the Riemann surface Σ inside the world-volume of the $N = \infty$ coincident Dp-branes.

As we have seen in Section 4.1, one can interpret the infinite-dimensional representation of $\pi_1(\Sigma)$ given in [18] as defining a non-commutative $C^{\star}$-algebra $C^{\star}(\Gamma, \theta)$. Through the Sen–Witten construction, the latter is the result of turning on a nonzero 't Hooft magnetic flux in the world-volume of the $N' = \infty$ Dp'-brane/antibrane pairs that are equivalent to $N = \infty$ coincident Dp-branes wrapped on Σ. Alternatively, through the anomaly cancellation condition, this flux is due to turning on a background Neveu–Schwarz B-field.

7.4. *Choice of the fibre bundle over Σ*

Given that $H^3(\Sigma, \mathbf{Z})$ is trivial, one would like to wrap the Dp-branes on a manifold whose real dimension is greater than 2. This would allow the correspondence between $\mathcal{K}$-bundles and classes in $H^3(\Sigma, \mathbf{Z})$ a possibility of being nontrivial. Fibre bundles over the Riemann surface Σ thus come to mind. We will not attempt a complete classification of all possibilities, as in fact trivial bundles over Σ will suffice. We will satisfy ourselves with an example of a trivial bundle $\Sigma \times Y$, for a certain choice of the space–time manifold Y, that will allow for a nontrivial torsion. Again it will turn out that more than one choice for Y is possible. The space–time manifold Y will be determined imposing consistency conditions.

In type IIB superstring theory, the manifold Y must be orientable and spin. Furthermore, $Q = \Sigma \times Y$ must allow for a nontrivial torsion subgroup within $H^3(Q, \mathbf{Z})$. Finally, $H^3(Q, \mathbf{Z})$ parameterises isomorphism classes of $\mathcal{K}$-bundles over Q, but instead we need it to parameterise isomorphism classes of $\mathcal{K}$-bundles over Σ. Hence Y must be chosen in such a way that torsion classes in $H^3(Q, \mathbf{Z})$ continue to parameterise $\mathcal{K}$-bundles over Σ.

This refines the minimum value of p determined in Section 5, where it was found that $p \geq 3$. A nontrivial $H^3(Q, \mathbf{Z})$ further imposes $p > 3$. Indeed, $p = 3$ would correspond to a $(1 + 1)$-dimensional Y. Factorise it (at least locally) as the product of a time-like factor Y_t times a space-like factor Y_x. The latter can be chosen compact or not, which leads to these topologically different choices for Y_x: S^1 and $\mathbf{R}$, and quotients thereof, such as $\mathbf{RP}^1$, for example. One finds that none of these choices satisfies our needs. Taking $Y_x = \mathbf{R}$ leads to a trivial $H^3(Q,\mathbf{Z})$. The choice $Y_x = S^1$, while producing a nontrivial $H^3(Q,\mathbf{Z})$, is torsionless; so is the case of $\mathbf{RP}^1$.

Within the type IIB theory the next allowed value is $p = 5$. Again separating out the trivial time-like dimension, let us see that one can find a space-like manifold Y in real dimension 3 satisfying the necessary requirements.

For the correspondence between torsion classes in $H^3(Q,\mathbf{Z})$ and $\mathcal{K}$-bundles over Σ to hold, one would on first sight require Y to have a trivial fundamental group, so that $\pi_1(Q) = \pi_1(\Sigma)$. However, this condition can be relaxed to a less stringent one. We will see presently that an abelian $\pi_1(Y)$ will suffice. Kunneth's formula [30] allows us to write

$$\begin{aligned} H^3(\Sigma\times Y,\mathbf{Z}) \subset{}& H^0(\Sigma,\mathbf{Z})\otimes H^3(\mathbf{Y},\mathbf{Z})\oplus H^1(\Sigma,\mathbf{Z})\otimes H^2(\mathbf{Y},\mathbf{Z})\oplus H^2(\Sigma,\mathbf{Z})\\ &\otimes H^1(\mathbf{Y},\mathbf{Z})\oplus H^3(\Sigma,\mathbf{Z})\otimes H^0(\mathbf{Y},\mathbf{Z}). \end{aligned} \tag{39}$$

In the particular case at hand, one can show that the above inclusion is actually an equality. Now $H^3(\Sigma,\mathbf{Z})$ is identically zero, while $H^0(\Sigma,\mathbf{Z}) = \mathbf{Z} = H^2(\Sigma,\mathbf{Z})$ and $H^1(\Sigma,\mathbf{Z}) = \mathbf{Z}^{2g}$. Torsion pieces, if any, must come from $H^3(Y,\mathbf{Z})$, $H^2(Y,\mathbf{Z})$ and $H^1(Y,\mathbf{Z})$. Allowing for an abelian $\pi_1(Y)$ for the moment, the manifold $\mathbf{RP}^3$ (which is orientable and spin) has a nontrivial torsion

$$H^1(\mathbf{RP}^3,\mathbf{Z}) = \mathbf{Z}_2. \tag{40}$$

More generally, branes on group manifolds have been studied in [34].

It remains to explain why one can allow for an abelian $\pi_1(Y)$ without spoiling the one-to-one correspondence between torsion classes in $H^3(Q,\mathbf{Z})$ and isomorphism classes of $\mathcal{K}$-bundles over Σ. The latter are in bijective correspondence with projectively unitary representations of $\pi_1(\Sigma)$. Now the decomposition $\pi_1(Q) = \pi_1(\Sigma)\times\pi_1(Y)$ together with Eq. (18) provides the answer: factors coming from an abelian $\pi_1(Y)$ will cancel when computing the left-hand side of (18). (We could even allow for a projectively represented abelian group $\pi_1(Y)$, at the cost of considering its nontrivial contribution to right-hand side of (18).)

We close this section with an observation. The anomaly cancellation condition is key to our construction. We have applied it within type IIB superstring theory, in order to link it to the Sen–Witten superposition of branes with antibranes. However, one could just as well apply it to bosonic string theory, where non-orientable manifolds are allowed and the anomaly cancellation condition [26] simplifies to

$$\beta_N([f]) = [H]_Q. \tag{41}$$

The requirements on the manifold Y thus become less restringent, and one can verify that the following examples satisfy all our needs. The two-dimensional real projective space $\mathbf{RP}^2$ and the Klein surface $\mathbf{K}^2$ have nontrivial torsion given by

$$H^1(\mathbf{RP}^2,\mathbf{Z}) = \mathbf{Z}_2,\qquad H^1(\mathbf{K}^2,\mathbf{Z}) = \mathbf{Z}\oplus\mathbf{Z}_2. \tag{42}$$

The absence of supersymmetry in our construction (see also Section 8) allows us to consider these possibilities as valid for the physical realisation of $C^{\star}(\Gamma, \theta)$ in terms of strings and branes.

8. BPS-like spectra in $g > 1$ from the Narasimhan–Seshadri theorem

In $g = 1$, Morita equivalence of non-commutative gauge theories is reflected in the T-duality of superstring theory [35]. If we were to follow the reasoning applied in $g = 1$ [20], we would now have to identify the dual tessellation $T^*(\mathbf{H})$. The latter would parameterise the endomorphisms End E_{NM} of the module E_{NM}. However, $T^*(\mathbf{H})$ must be a quantum space, since Γ is non-abelian. Moreover, in $g > 1$ there is no T-duality, and compactification breaks all supersymmetry [33]. Hence, unlike in $g = 1$, there are no supersymmetric BPS spectra in $g > 1$. This notwithstanding, the breakdown of supersymmetry does not prevent the existence of stable, non-BPS states in M-theory [2,10].

We will therefore follow an alternative route. We will prove the existence of constant-curvature connections on the projective modules E_{NM}. We will see that, as in $g = 1$, in $g > 1$ there exists a moduli space of such connections. Even though there is no supersymmetry, one can take such connections as defining the $g > 1$ analogues of BPS states on the torus, since the latter were also characterised as having constant-curvature. In $g = 1$ the stability of such states was a consequence of supersymmetry. In the absence of supersymmetry, however, the stability of these states deserves a separate study.

8.1. Constant-curvature connections on E_{NM}

The finite-dimensional space $\mathbf{C}^N_{(M)}$ in Eq. (23) is the fibre of a stable holomorphic bundle over Σ. Let us assume that the double-scaling limit $M \to -\infty$, $N \to \infty$ respects stability. In other words, we assume that this limit can be taken in such a way that $L^2(\mathbf{H})$ becomes the fibre of an (infinite-dimensional) stable holomorphic bundle over Σ. Then a suitable infinite-dimensional generalisation of Donaldson's version of the Narasimhan–Seshadri theorem establishes the existence of a metric connection ∇_D such that the constant-curvature condition (16)

$$F_D = -2\pi \mathrm{i} \left(\frac{M}{N} - \theta_b \right) \omega \mathbf{I} \tag{43}$$

holds. Above, ω equals the Poincaré 2-form $\mathrm{d}z \wedge \mathrm{d}\bar{z}/(\mathrm{Im}\, z)^2$ on $\mathbf{H}$, and $\mathbf{I}$ denotes the identity on E_{NM}.

A remark is in order. There is a formal analogy between Eq. (3) and Eq. (43). However, contrary to the non-commutative torus, our non-commutative $C^{\star}$-algebra $C^{\star}(\Gamma, \theta)$ and its projective modules cannot be obtained from the representation theory of the Heisenberg algebra. In fact we have followed a route different from that of the non-commutative torus [20]. In $g = 1$ one first constructs a derivation δ of the $C^{\star}$-algebra. Next one uses δ in order to define a connection ∇. Finally, ∇ is used, as in Eq. (3), in order to impose the constant-curvature condition. In $g > 1$ we have bypassed this procedure because the constant-curvature condition (43) is no longer a Heisenberg algebra. Without defining a derivation δ of $C^{\star}(\Gamma, \theta)$, the Narasimhan–Seshadri theorem directly allows us to construct the desired connections on the projective modules E_{NM}.

8.2. Moduli space of constant-curvature connections

The previous construction relied on the notion of stability for holomorphic vector bundles over Σ. As we have seen, stability is required in order to have constant-curvature connections or, in physical terms, BPS-like states. There is one more reason to require stability. In $g = 1$ there exists a moduli space of BPS states. Does a moduli space of BPS-like states exist in $g > 1$?

Topological vector bundles over Σ are classified, up to isomorphism, by the rank N and the degree M. However, the classification of *holomorphic* vector bundles involves continuous parameters, and so we have a moduli space of holomorphic vector bundles over Σ. From the above it follows that this moduli space coincides with that of constant-curvature connections. The latter define the higher-genus analogue of BPS states. So the $g > 1$ analogues of BPS states are parameterised by the points of the moduli space of holomorphic vector bundles. It turns out that the latter space in general is not Hausdorff, but the condition of stability suffices to ensure a good moduli space. The precise statement is as follows [36]: fix the data Σ, N and M, the latter two coprime. Then there exists a complex smooth, connected and compact moduli space $\mathcal{M}_{NM}^{(g)}$ of equivalence classes of rank N, degree M, stable holomorphic vector bundles over Σ, with dimension $N^2(g-1)+1$. The moduli space $\mathcal{M}_{NM}^{(g)}$ depends only on the residue class of M modulo N.

9. Conclusions and outlook

In this paper, we have established an interesting link between non-commutative geometry and the Sen–Witten construction of non-BPS branes, by explicitly constructing a non-commutative $C^{\star}$-algebra $C^{\star}(\Gamma, \theta)$ that generalises to $g > 1$ what the non-commutative torus does in $g = 1$. The mathematical definition of $C^{\star}(\Gamma, \theta)$ was presented in [18]; in this paper it has been given a physical interpretation in terms of the wrapping of Dp-branes on a Riemann surface Σ in $g > 1$, with a background B-field turned on. The latter deforms the commutative $C^{\star}$-algebra of functions to a non-commutative $C^{\star}$-algebra that we have succeeded in identifying. Finally, we have constructed a family of projective modules over $C^{\star}(\Gamma, \theta)$ and proved the existence of constant-curvature connections on them.

In $g = 1$, Morita equivalence led to a whole $SL_2(\mathbf{Z})$ orbit of Morita-equivalent non-commutative tori [20]. This was due to the abelian property of the fundamental group of the torus, which allowed for an easy identification of the commutant. However, the fact that the Fuchsian group uniformising a Riemann surface in $g > 1$ is non-abelian implies that there exists no Morita-group orbit of $C^{\star}(\tilde{\Gamma}, \tilde{\theta})$ algebras that are Morita-equivalent to $C^{\star}(\Gamma, \theta)$. This notwithstanding, we have succeeded in identifying the $g > 1$ analogues of supersymmetric BPS states on the non-commutative torus, thanks to Donaldson's description of stable vector bundles over Riemann surfaces.

An important physical question to address in this context is the stability of the BPS-like states constructed here. It would be very interesting to relate the mathematical property of stability of holomorphic vector bundles with the physical property of being a stable, non-BPS state. Mathematically, one would like to compute the topological numbers and the Chern character for the projective $C^{\star}$-modules E_{NM}. We hope to report on these issues in the future.

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